

Lost in Abstraction: Monotonicity in Multi-Threaded Programs (Extended Technical Report)*

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Abstract. *Monotonicity* in concurrent systems stipulates that, in any global state, extant system actions remain executable when new processes are added to the state. This concept is not only natural and common in multi-threaded software, but also useful: if every thread’s memory is finite, monotonicity often guarantees the decidability of safety property verification even when the number of running threads is unknown. In this paper, we show that the act of obtaining finite-data thread abstractions for model checking can be at odds with monotonicity: Predicate-abstraction certain widely used monotone software results in non-monotone multi-threaded Boolean programs — the monotonicity is *lost in the abstraction*. As a result, well-established sound and complete safety checking algorithms become inapplicable; in fact, safety checking turns out to be undecidable for the obtained class of unbounded-thread Boolean programs. We demonstrate how the abstract programs can be modified into monotone ones, without affecting safety properties of the non-monotone abstraction. This significantly improves earlier approaches of enforcing monotonicity via overapproximations.

1 Introduction

This paper addresses non-recursive procedures executed by multiple threads (e.g. dynamically generated, and possibly unbounded in number), which communicate via shared variables or higher-level mechanisms such as mutexes. OS-level code, including Windows, UNIX, and Mac OS device drivers, makes frequent use of such concurrency APIs, whose correct use is therefore critical to ensure a reliable programming environment.

The utility of *predicate abstraction* as a safety analysis method is known to depend critically on the choice of predicates: the consequences of a poor choice range from inferior performance to flat-out unprovability of certain properties. We propose in this paper an extension of predicate abstraction to multi-threaded programs that enables reasoning about intricate data relationships, namely

shared-variable: “shared variables **s** and **t** are equal”,

*This work is supported by the Toyota Motor Corporation, NSF grant no. 1253331 and ERC project 280053.

single-thread: “local variable l of thread i is less than shared variable s ”, and
inter-thread: “local variable l of thread i is less than variable l in all other threads”.

Why such a rich predicate language? For certain concurrent algorithms such as the widely used *ticket* busy-wait lock algorithm [4] (the default locking mechanism in the Linux kernel since 2008; see Fig. 1), the verification of elementary safety properties **requires** single- and inter-thread relationships. They are needed to express, for instance, that a thread holds the minimum ticket value, an inter-thread relationship.

In the main part of the paper, we address the problem of full parameterized (unbounded-thread) program verification with respect to our rich predicate language. Such reasoning requires first that the n -thread abstract program $\hat{\mathcal{P}}^n$, obtained by existential inter-thread predicate abstraction of the n -thread concrete program \mathcal{P}^n , is rewritten into a single template program $\tilde{\mathcal{P}}$ to be executed by (any number of) multiple threads. In order to capture the semantics of these programs in the template $\tilde{\mathcal{P}}$, the template programming language must itself permit variables that refer to the currently executing or a generic passive thread; we call such programs *dual-reference (DR)*. We describe how to obtain $\tilde{\mathcal{P}}$, namely essentially as an overapproximation of $\hat{\mathcal{P}}^b$, for a constant b that scales linearly with the number of inter-thread predicates used in the predicate abstraction.

Given the *Boolean* dual-reference program $\tilde{\mathcal{P}}$, we might now expect the unbounded-thread replicated program $\tilde{\mathcal{P}}^\infty$ to form a classical *well quasi-ordered transition system* [2], enabling the fully automated, algorithmic safety property verification in the abstract. This turns out not to be the case: the expressiveness of dual-reference programs renders parameterized program location reachability undecidable, despite the finite-domain variables. The root cause is the lack of *monotonicity* of the transition relation with respect to the standard partial order over the space of unbounded thread counters. That is, adding passive threads to the source state of a valid transition can invalidate this transition and in fact block the system. Since the input C programs are, by contrast, perfectly monotone, we say the monotonicity is *lost in the abstraction*. As a result, our abstract programs are in fact not well quasi-ordered.

Inspired by earlier work on *monotonic abstractions* [3], we address this problem by restoring the monotonicity using a simple *closure operator*, which enriches the transition relation of the abstract program $\tilde{\mathcal{P}}$ such that the obtained program $\tilde{\mathcal{P}}_m$ engenders a monotone (and thus well quasi-ordered) system. The closure operator essentially terminates passive threads that block transitions allowed by other passive threads. In contrast to those earlier approaches, which *enforce* (rather than *restore*) monotonicity in genuinely non-monotone systems, we exploit the fact that the input programs are monotone. As a result, the monotonicity closure $\tilde{\mathcal{P}}_m$ can be shown to be *safety-equivalent* to the intermediate program $\tilde{\mathcal{P}}$.

To summarize, the central contribution of this paper is a predicate abstraction strategy for unbounded-thread C programs, with respect to the rich language of inter-thread predicates. This language allows the abstraction to track

```

struct Spinlock {
    natural s := 1; // ticket being served
    natural t := 1; // next free ticket

    struct Spinlock lock; // shared

    void spin_lock() {
        natural l := 0; // local
         $\ell_1$ : l := fetch_and_add(lock.t);
         $\ell_2$ : while (l ≠ lock.s)
            /* spin */; }

    void spin_unlock() {
         $\ell_3$ : lock.s++; }

```

The ticket algorithm: Shared variable *lock* has two integer components: *s* holds the ticket currently served (or, if none, the ticket served next), while *t* holds the ticket to be served after all waiting threads have had access.

To request access to the locked region, a thread atomically retrieves the value of *t* and then increments *t*. The thread then busy-waits (“spins”) until local variable *l* agrees with shared *s*. To unlock, a thread increments *s*.

See App. A for more intuition.

Fig. 1: Our goal is to verify “unbounded-thread mutual exclusion”: no matter how many threads try to acquire and release the lock concurrently, no two of them should simultaneously be between the calls to functions `spin_lock` and `spin_unlock`.

properties that are essentially universally quantified over all passive threads. To this end, we first develop such a strategy for a fixed number of threads. Second, in preparation for extending it to the unbounded case, we describe how the abstract model, obtained by existential predicate abstraction for a given thread count *n*, can be expressed as a template program that can be multiply instantiated. Third, we show a sound and complete algorithm for reachability analysis for the obtained parameterized Boolean dual-reference programs. We overcome the undecidability of the problem by building a monotone closure that enjoys the same safety properties as the original abstract dual-reference program.

We omit in this submission practical aspects such as predicate discovery, the algorithmic construction of the abstract programs, and abstraction refinement. We provide, however, an extensive appendix, with proofs of all lemmas and theorems.

2 Inter-Thread Predicate Abstraction

In this section we introduce single- and inter-thread predicates, with respect to which we then formalize existential predicate abstraction. Except for the extended predicate language, these concepts are mostly standard and lay the technical foundations for the contributions of this paper.

2.1 Input Programs and Predicate Language

2.1.1 Asynchronous Programs An *asynchronous program* \mathcal{P} allows only one thread at a time to change its local state. We model \mathcal{P} , designed for execution

by $n \geq 1$ concurrent threads, as follows. The variable set V of a program \mathcal{P} is partitioned into sets S and L . The variables in S , called *shared*, are accessible jointly by all threads, and those in L , called *local*, are accessible by the individual thread that owns the variable. We assume the statements of \mathcal{P} are given by a transition formula \mathcal{R} over unprimed (current-state) and primed (next-state) variables, V and $V' = \{v' : v \in V\}$. Further, the initial states are characterized by the initial formula \mathcal{I} over V . We assume \mathcal{I} is expressible in a suitable logic for which existential quantification is computable (required later for the abstraction step).

As usual, the computation may be controlled by a local program counter pc , and involve non-recursive function calls. When executed by n threads, \mathcal{P} gives rise to *n-thread program states* consisting of the valuations of the variables in $V_n = S \cup L_1 \cup \dots \cup L_n$, where $L_i = \{l_i : l \in L\}$. We call a variable set *uniformly indexed* if its variables either all have no index, or all have the same index. For a formula f and two uniformly-indexed variable sets X_1 and X_2 , let $f\{X_1 \triangleright X_2\}$ denote f after replacing every occurrence of a variable in X_1 by the variable in X_2 with the same base name, if any; unreplaced if none. We write $f\{X_1 \bowtie X_2\}$ short for $f\{X_1 \triangleright X_2\}\{X_1' \triangleright X_2'\}$. As an example, given $S = \{s\}$ and $L = \{l\}$, we have $(l' = l + s)\{L \bowtie L_a\} = (l'_a = l_a + s)$. Finally, let $X \stackrel{\circ}{=} X'$ stand for $\forall x \in X : x = x'$.

The *n-thread instantiation* \mathcal{P}^n is defined for $n \geq 1$ as

$$\mathcal{P}^n = (\mathcal{R}^n, \mathcal{I}^n) = \left(\bigvee_{a=1}^n (\mathcal{R}_a)^n, \bigwedge_{a=1}^n \mathcal{I}\{L \triangleright L_a\} \right) \quad (1)$$

where

$$(\mathcal{R}_a)^n :: \mathcal{R}\{L \bowtie L_a\} \wedge \bigwedge_{p: p \neq a} L_p \stackrel{\circ}{=} L'_p. \quad (2)$$

Formula $(\mathcal{R}_a)^n$ asserts that the shared variables, and the variables of the *active* (executing) thread a are updated according to \mathcal{R} , while the local variables of passive threads $p \neq a$ are not modified (p ranges over $\{1, \dots, n\}$). A state is *initial* if all threads are in a state satisfying \mathcal{I} . An *n-thread execution* is a sequence of *n-thread program states* whose first state satisfies \mathcal{I}^n and whose consecutive states are related by \mathcal{R}^n . We assume the existence of an error location in \mathcal{P} ; an *error state* is one where some thread resides in the error location. \mathcal{P} is *safe* if no execution exists that ends in an error state. Mutex conditions can be checked using a ghost semaphore and redirecting threads to the error location if they try to access the critical section while the semaphore is set.

2.1.2 Predicate Language We extend the predicate language from [10] to allow the use of the *passive-thread variables* $L_P = \{l_P : l \in L\}$, each of which represents a local variable owned by a generic passive thread. The presence of variables of various categories gives rise to the following predicate classification.

Definition 1 A predicate Q over S , L and L_P is **shared** if it contains variables from S only, **local** if it contains variables from L only, **single-thread** if it contains variables from L but not from L_P , and **inter-thread** if it contains variables from L and from L_P .

Single- and inter-thread prediactes may contain variables from S . For example, in the ticket algorithm (Fig. 1), with $S = \{\mathbf{s}, \mathbf{t}\}$ and $L = \{1\}$, examples of shared, local, single- and inter-thread predicates are: $\mathbf{s} = \mathbf{t}$, $1 = 5$, $\mathbf{s} = 1$ and $1 \neq 1_P$, respectively.

Semantics Let $Q[1], \dots, Q[m]$ be m predicates (any class). Predicate $Q[i]$ is evaluated in a given n -thread state v ($n \geq 2$) with respect to a choice of active thread a :

$$Q[i]_a \quad :: \quad \bigwedge_{p:p \neq a} Q[i]\{L \triangleright L_a\}\{L_P \triangleright L_p\}. \quad (3)$$

As special cases, for single-thread and shared predicates (no L_P variables), we have $Q[i]_a = Q[i]\{L \triangleright L_a\}$ and $Q[i]_a = Q[i]$, resp. We write $v \models Q[i]_a$ if $Q[i]_a$ holds in state v . Predicates $Q[i]$ give rise to an abstraction function α , mapping each n -thread program state v to an $m \times n$ bit matrix with entries

$$\alpha(v)_{i,a} = \begin{cases} \mathbf{T} & \text{if } v \models Q[i]_a \\ \mathbf{F} & \text{otherwise.} \end{cases} \quad (4)$$

Function α partitions the n -thread program state space via m predicates into $2^{m \times n}$ equivalence classes. As an example, consider the inter-thread predicates $1 \leq 1_P$, $1 > 1_P$, and $1 \neq 1_P$ for a local variable 1, $n = 4$ and the state $v :: (1_1, 1_2, 1_3, 1_4) = (4, 4, 5, 6)$:

$$\alpha(v) = \begin{pmatrix} \mathbf{T} & \mathbf{T} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{T} \\ \mathbf{F} & \mathbf{F} & \mathbf{T} & \mathbf{T} \end{pmatrix}. \quad (5)$$

In the matrix, row $i \in \{1, 2, 3\}$ lists the truth of predicate $Q[i]$ for each of the four threads in the active role. Predicate $1 \leq 1_P$ captures whether a thread owns the minimum value for local variable 1 (true for $a = 1, 2$); $1 > 1_P$ tracks whether a thread owns the *unique* maximum value (true for $a = 4$); finally $1 \neq 1_P$ captures the uniqueness of a thread's copy of 1 (true for $a = 3, 4$).

Inter-thread predicates and abstraction Predicates that reason universally about threads have been used successfully as targets in (inductive) invariant generation procedures [5, 25]. In this paper we discuss their role in abstractions. The use of these fairly expressive and presumably expensive predicates is not by chance: automated methods that cannot reason about them [13, 10, 28] essentially fail for the ticket algorithm in Fig. 1: for a fixed number of threads that concurrently and repeatedly (e.g. in an infinite loop) request and release lock ownership, the inter-thread relationships need to be “simulated” via enumeration, incurring very high time and space requirements, even for a handful of threads. In the unbounded-thread case, they diverge. This is essentially due to known limits of thread-modular and Owicki-Gries style proof systems, which do not have access to inter-thread predicates [23]. App. A shows that the number of *single-thread* predicates needed to prove correctness of the ticket algorithm depends on n , from which unprovability in the unbounded case follows.

2.2 Existential Inter-Thread Predicate Abstraction

Embedded into our formalism, the goal of *existential predicate abstraction* [8, 18] is to derive an abstract program $\hat{\mathcal{P}}^n$ by treating the equivalence classes induced by Eq. (4) as abstract states. $\hat{\mathcal{P}}^n$ thus has $m \times n$ Boolean variables:

$$\hat{V}_n = \bigcup_{a=1}^n \hat{L}_a = \bigcup_{a=1}^n \{b[i]_a : 1 \leq i \leq m\}.$$

Variable $b[i]_a$ tracks the truth of predicate $Q[i]$ for active thread a . This is formalized in (6), relating concrete and abstract n -thread states (valuations of V_n and \hat{V}_n , resp.):

$$\mathcal{D}^n :: \bigwedge_{i=1}^m \bigwedge_{a=1}^n b[i]_a \Leftrightarrow Q[i]_a. \quad (6)$$

For a formula f , let f' denote f after replacing each variable by its primed version. We then have $\hat{\mathcal{P}}^n = (\hat{\mathcal{R}}^n, \hat{\mathcal{I}}^n) = \left(\bigvee_{a=1}^n (\hat{\mathcal{R}}_a)^n, \hat{\mathcal{I}}^n \right)$ where

$$(\hat{\mathcal{R}}_a)^n :: \exists V_n V'_n : (\mathcal{R}_a)^n \wedge \mathcal{D}^n \wedge (\mathcal{D}^n)', \quad (7)$$

$$\hat{\mathcal{I}}^n :: \exists V_n : \mathcal{I}^n \wedge \mathcal{D}^n. \quad (8)$$

As an example, consider the decrement operation $1 := 1 - 1$ on a local integer variable 1 , and the inter-thread predicate $1 < 1_P$. Using Eq. (7) with $n = 2$, $a = 1$, we get 4 abstract transitions, which are listed in Table 1. The table shows that the abstraction is no longer asynchronous (treating b_1 as belonging to thread 1, b_2 to thread 2): in the highlighted transition, the executing thread 1 changes (its pc and hence) its local state, and so does thread 2. By contrast, on the right we have $1_2 = 1'_2$ in all rows. The loss of asynchrony will become relevant in Sect. 3, where we define a suitable abstract Boolean programming language (which then necessarily must accommodate non-asynchronous programs).

b_1	b_2	b'_1	b'_2	1_1	1_2	$1'_1$	$1'_2$
F	F	T	F	1	1	0	1
F	T	F	F	1	0	0	0
F	T	F	T	2	0	1	0
T	F	T	F	1	2	0	2

Table 1: Abstraction $(\hat{\mathcal{R}}_1)^2$ for stmt. $1 := 1 - 1$ against predicate $1 < 1_P$ (left); concrete witness transitions, i.e. elements of $(\mathcal{R}_1)^2$ (right). The highlighted row indicates asynchrony violations

Proving the ticket algorithm (fixed-thread case) As in any existential abstraction, the abstract program $\hat{\mathcal{P}}^n$ overapproximates (the set of executions of) the concrete program \mathcal{P}^n ; the former can therefore be used to verify safety of the latter. We illustrate this using the ticket algorithm (Fig. 1). Consider the predicates

$Q[1] :: 1 \neq 1_P$, $Q[2] :: t > \max(1, 1_P)$, and $Q[3] :: s = 1$. The first two are inter-thread; the third is single-thread. The predicates assert the uniqueness of a ticket ($Q[1]$), that the next free ticket is larger than all tickets currently owned by threads ($Q[2]$), and that a thread's ticket is currently being served ($Q[3]$). The abstract reachability tree for $\hat{\mathcal{P}}^n$ and these predicates reveals that mutual exclusion is satisfied: there is no state with both threads in location ℓ_3 . The tree grows exponentially with n .

3 From Existential to Parametric Abstraction

Classical existential abstraction as described in Sect. 2.2 obliterates the symmetry present in the concrete concurrent program, which is given as the n -thread instantiation of a single-thread template \mathcal{P} : the abstraction is instead formulated via predicates over the explicitly expanded n -thread program \mathcal{R}^n . As observed in previous work [10], such a “symmetry-oblivious” approach suffers from poor scalability for fixed-thread verification problems. Moreover, *parametric* reasoning over an unknown number of threads is impossible since the abstraction (7) directly depends on n .

To overcome these problems, we now derive an overapproximation of $\hat{\mathcal{P}}^n$ via a generic program template $\tilde{\mathcal{P}}$ that can be instantiated for any n . There is, however, one obstacle: instantiating a program (such as \mathcal{P}) formulated over shared variables and one copy of the thread-local variables naturally gives rise to asynchronous concurrency. The programs resulting from inter-thread predicate abstraction are, however, not asynchronous, as we have seen. As a result, we need a more powerful abstract programming language.

3.1 Dual-Reference Programs

In contrast to asynchronous programs, the variable set \tilde{V} of a *dual-reference (DR)* program $\tilde{\mathcal{P}}$ is partitioned into two sets: \tilde{L} , the local variables of the active thread as before, and $\tilde{L}_P = \{1_P : 1 \in \tilde{L}\}$. The latter set contains passive-thread variables, which, intuitively, regulate the behavior of non-executing threads. To simplify reasoning about DR programs, we exclude classical shared variables from the description: they can be simulated using the active and passive flavors of local variables (see App. B).

The statements of $\tilde{\mathcal{P}}$ are given by a transition formula $\tilde{\mathcal{R}}$ over \tilde{V} and \tilde{V}' , now potentially including passive-thread variables. Similarly, $\tilde{\mathcal{I}}$ may contain variables from \tilde{L}_P . The n -thread instantiation $\tilde{\mathcal{P}}^n$ of a DR program $\tilde{\mathcal{P}}$ is defined for $n \geq 2$ as

$$\tilde{\mathcal{P}}^n = (\tilde{\mathcal{R}}^n, \tilde{\mathcal{I}}^n) = \left(\bigvee_{a=1}^n (\tilde{\mathcal{R}}_a)^n, \bigvee_{a=1}^n (\tilde{\mathcal{I}}_a)^n \right) \quad (9)$$

where

$$(\tilde{\mathcal{R}}_a)^n :: \bigwedge_{p:p \neq a} \tilde{\mathcal{R}}\{\tilde{L} \bowtie \tilde{L}_a\} \{\tilde{L}_P \bowtie \tilde{L}_p\} \quad (10)$$

$$(\tilde{\mathcal{I}}_a)^n :: \bigwedge_{p:p \neq a} \tilde{\mathcal{I}}\{\tilde{L} \triangleright \tilde{L}_a\} \{\tilde{L}_P \triangleright \tilde{L}_p\} \quad (11)$$

Recall that $f\{X_1 \bowtie X_2\}$ denotes index replacement of both current-state and next-state variables. Eq. (10) encodes the effect of a transition on the active thread a , and $n-1$ passive threads p . The conjunction ensures that the transition formula $\tilde{\mathcal{R}}$ holds no matter which thread $p \neq a$ takes the role of the passive thread: transitions that “work” only for select passive threads are rejected.

3.2 Computing an Abstract Dual-Reference Template

From the existential abstraction $\hat{\mathcal{P}}^n$ we derive a Boolean dual-reference template program $\tilde{\mathcal{P}}$ such that, for all n , the n -fold instantiation $\tilde{\mathcal{P}}^n$ overapproximates $\hat{\mathcal{P}}^n$. The variables of $\tilde{\mathcal{P}}$ are $\tilde{L} = \{b[i] : 1 \leq i \leq m\}$ and $\tilde{L}_P = \{b[i]_P : 1 \leq i \leq m\}$. Intuitively, the transitions of $\tilde{\mathcal{P}}$ are those that are feasible, for **some** n , in $\hat{\mathcal{P}}^n$, given active thread 1 and passive thread 2. We first compute the set $\tilde{\mathcal{R}}(n)$ of these transitions for fixed n . Formally, the components of $\tilde{\mathcal{P}}(n) = (\tilde{\mathcal{R}}(n), \tilde{\mathcal{I}}(n))$ are, for $n \geq 2$,

$$\tilde{\mathcal{R}}(n) :: \exists \hat{L}_3, \hat{L}'_3, \dots, \hat{L}_n, \hat{L}'_n : (\hat{\mathcal{R}}_1)^n \{ \hat{L}_1 \bowtie \tilde{L} \} \{ \hat{L}_2 \bowtie \tilde{L}_P \} \quad (12)$$

$$\tilde{\mathcal{I}}(n) :: \exists \hat{L}_3, \dots, \hat{L}_n : \hat{\mathcal{I}}^n \{ \hat{L}_1 \triangleright \tilde{L} \} \{ \hat{L}_2 \triangleright \tilde{L}_P \} \quad (13)$$

We apply this strategy to the earlier example of the decrement statement $1 := 1-1$. To compute Eq. (12) first with $n = 2$, we need $(\hat{\mathcal{R}}_1)^2$, which was enumerated previously in Table 1. Simplification results in a Boolean DR program with variables \mathbf{b} and \mathbf{b}_P and transition relation

$$\tilde{\mathcal{R}}(2) = (\neg \mathbf{b} \wedge \mathbf{b}_P \wedge \neg \mathbf{b}') \vee (\neg \mathbf{b}_P \wedge \mathbf{b}' \wedge \neg \mathbf{b}'_P). \quad (14)$$

Using (14) as the template $\tilde{\mathcal{R}}$ in (10) generates existential abstractions of many concrete decrement transitions; for instance, for $n = 2$ and $a = 1$ we get back the transition relation in Table 1. The question is now: does (14) suffice as a template, i.e. does $(\tilde{\mathcal{R}}(2))^n$ overapproximate $\hat{\mathcal{R}}^n$ for all n ? The answer is no: the abstract 3-thread transitions shown in Table 2 are not permitted by $(\tilde{\mathcal{R}}(2))^n$ for any n , since neither $\neg \mathbf{b} \wedge \mathbf{b}_P$ nor $\mathbf{b}' \wedge \neg \mathbf{b}'_P$ are satisfied for all choices of passive threads (violations highlighted in the table).

We thus increase n to 3, recompute Eq. (12), and obtain

$$\tilde{\mathcal{R}}(3) :: \tilde{\mathcal{R}}(2) \vee (\neg \mathbf{b} \wedge \neg \mathbf{b}_P \wedge \neg \mathbf{b}' \wedge \neg \mathbf{b}'_P). \quad (15)$$

The new disjunct accommodates the abstract transitions highlighted in Table 2, which were missing before.

Does $(\tilde{\mathcal{R}}(3))^n$ overapproximate $\hat{\mathcal{R}}^n$ for all n ? When does the process of increasing n stop? To answer these questions, we first state the following diagonalization lemma, which helps us prove the overapproximation property for the template program.

Lemma 2 $(\tilde{\mathcal{P}}(n))^n$ overapproximates $\hat{\mathcal{P}}^n$: For every $n \geq 2$ and every a , $(\hat{\mathcal{R}}_a)^n \Rightarrow (\tilde{\mathcal{R}}(n)_a)^n$ and $\hat{\mathcal{I}}^n \Rightarrow (\tilde{\mathcal{I}}(n)_a)^n$.

b_1	b_2	b_3	b'_1	b'_2	b'_3	l_1	l_2	l_3	l'_1	l'_2	l'_3
F	F	F	F	F	F	1	0	0	0	0	0
F	F	T	F	F	F	1	1	0	0	1	0
F	F	T	F	F	T	2	1	0	1	1	0

Table 2: Part of the abstraction $(\tilde{\mathcal{R}}_1)^3$ for stmt. $l := l - 1$ against predicate $l < l_P$ (left); concrete witness transitions (right). The highlighted elements are inconsistent with (14) as a template

We finally give a saturation bound for the sequence $(\tilde{\mathcal{P}}(n))$. Along with the diagonalization lemma, this allows us to obtain a template program $\tilde{\mathcal{P}}$ independent of n , and enable parametric reasoning in the abstract.

Theorem 3 *Let $\#_{IT}$ be the number of inter-thread predicates among the $Q[i]$. Then the sequence $(\tilde{\mathcal{P}}(n))$ stabilizes at $\mathbf{b} = 4 \times \#_{IT} + 2$, i.e. for $n \geq \mathbf{b}$, $\tilde{\mathcal{P}}(n) = \tilde{\mathcal{P}}(\mathbf{b})$.*

Corollary 4 (from L. 2, T. 3) *Let $\tilde{\mathcal{P}} := \tilde{\mathcal{P}}(\mathbf{b})$, for \mathbf{b} as in Thm. 3. The components of $\tilde{\mathcal{P}}$ are thus $(\tilde{\mathcal{R}}, \tilde{\mathcal{I}}) = (\tilde{\mathcal{R}}(\mathbf{b}), \tilde{\mathcal{I}}(\mathbf{b}))$. Then, for $n \geq 2$, $\tilde{\mathcal{P}}^n$ overapproximates $\hat{\mathcal{P}}^n$.*

Building a template DR program thus requires instantiating the existentially abstracted transition relation for a number \mathbf{b} of threads that is linear in the number of inter-thread predicates with respect to which to abstraction is built.

As a consequence of losing asynchrony in the abstraction, many existing model checkers for concurrent software become inapplicable [27, 11, 12]. For a fixed thread count n , the problem can be circumvented by forgoing the replicated nature of the concurrent programs, as done in [10] for **boom** tool: it proves the ticket algorithm correct up to $n = 3$, but takes a disappointing 30 minutes. The goal of the following section is to design an efficient and, more importantly, fully parametric solution.

4 Unbounded-Thread Dual-Reference Programs

The multi-threaded Boolean dual-reference programs $\tilde{\mathcal{P}}^n$ resulting from predicate-abstraction asynchronous programs against inter-thread predicates are symmetric and free of recursion. The symmetry can be exploited using classical methods that “counterize” the state space [17]: a global state is encoded as a vector of local-state counters, each of which records the number of threads currently occupying a particular local state.

These methods are applicable to unbounded thread numbers as well, in which case the local state counters range over unbounded natural numbers $[0, \infty[$. The fact that the abstract program executed by each thread is finite-state now might suggest that the resulting infinite-state counter systems can be modeled as vector addition systems (as done in [17]) or, more generally, as *well quasi-ordered*

transition systems [15, 1] (defined below). This would give rise to sound and complete algorithms for local-state reachability in such programs.

This strategy turns out to be wrong: the full class of Boolean DR programs is expressive enough to render safety checking for an unbounded number of threads undecidable, despite the finite-domain variables:

Theorem 5 *Program location reachability for Boolean DR programs run by an unbounded number of threads is undecidable.*

The proof reduces the halting problem for 2-counter machines to a reachability problem for a DR program $\tilde{\mathcal{P}}$. Counter values c_i are reduced to numbers of threads in program locations d_i of $\tilde{\mathcal{P}}$. A zero-test for counter c_i is reduced to testing the *absence of any thread* in location d_i . This condition can be expressed using passive-thread variables, but not using traditional single-thread local variables. (Details of the proof in App. E.)

Thm. 5 implies that the unbounded-counter systems obtained from asynchronous programs are in fact *not* well quasi-ordered. How come? Can this problem be fixed, in order to permit a complete verification method? If so, at what cost?

4.1 Monotonicity in Dual-Reference Programs

For a transition system (Σ, \mapsto) to be well-quasi ordered, we need two conditions to be in place [15, 1, 2]:

well quasi-orderedness: there exists a reflexive and transitive binary relation \preceq on Σ such that for every infinite sequence v, w, \dots of states in Σ there exist i, j with $i < j$ and $v_i \preceq v_j$.

monotonicity: for any v, v', w with $v \mapsto v'$ and $v \preceq w$ there exists w' such that $w \mapsto w'$ and $v' \preceq w'$.

We apply this definition to the case of dual-reference programs. Representing global states of the abstract system $\tilde{\mathcal{P}}^n$ defined in Sect. 3 as counter tuples, we can define \preceq as

$$(n_1, \dots, n_k) \preceq (n'_1, \dots, n'_k) \quad :: \quad \forall i = 1..k : n_i \leq n'_i$$

where k is the number of thread-local states. We can now characterize monotonicity of DR programs as follows:

Lemma 6 *Let $\tilde{\mathcal{R}}$ be the transition relation of a DR program. Then the infinite-state transition system $\cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$ is monotone (with respect to \preceq) exactly if, for all $k \geq 2$:*

$$(v, v') \in \tilde{\mathcal{R}}^k \quad \Rightarrow \quad \forall l_{k+1} \exists l'_{k+1}, \pi : (\langle v, l_{k+1} \rangle, \pi(\langle v', l'_{k+1} \rangle)) \in \tilde{\mathcal{R}}^{k+1} . \quad (16)$$

In (16), the expression $\forall l_{k+1} \exists l'_{k+1} \dots$ quantifies over valuations of the local variables of thread $k+1$. The notation $\langle v, l_{k+1} \rangle$ denotes a $(k+1)$ -thread state that agrees with v in the first k local states and whose last local state is l_{k+1} ; similarly $\langle v', l'_{k+1} \rangle$. Symbol π denotes a permutation on $\{1, \dots, k+1\}$ that acts on states by acting on thread indices, which effectively reorders thread local states.

Asynchronous programs are trivially monotone (and DR): Eq. (16) is satisfied by choosing $l'_{k+1} := l_{k+1}$ and π the identity. Table 3 shows instructions found in *non*-asynchronous programs that destroy monotonicity, and why. For example, the swap instruction in the first row gives rise to a DR program with a 2-thread transition $(0, 0, 0, 0) \in \tilde{\mathcal{R}}^2$. Choosing $l_3 = 1$ in (16) requires the existence of a transition in $\tilde{\mathcal{R}}^3$ of the form $(l_1, l_2, l_3, l'_1, l'_2, l'_3) = (0, 0, 1, \pi(0, 0, l'_3))$, which is impossible: by equations (9) and (10), there must exist $a \in \{1, 2, 3\}$ such that for $\{p, q\} = \{1, 2, 3\} \setminus \{a\}$, both “ a swaps with p ” and “ a swaps with q ” hold, i.e.

$$l'_p = l_a \wedge l'_a = l_p \quad \wedge \quad l'_q = l_a \wedge l'_a = l_q,$$

which is equivalent to $l'_a = l_p = l_q \wedge l_a = l'_p = l'_q$. It is easy to see that this formula is inconsistent with the partial assignment $(0, 0, 1, \pi(0, 0, l'_3))$, no matter what l'_3 .

Dual-reference program		Monotonicity	
instruction	variables	mon.?	assgn. violating (17)
$l, l_P := l_P, l$	$l \in \mathbb{B}$	no	$l = 0, l' = 1$
$l, l_P := l + 1, l_P - 1$	$l \in \mathbb{N}$	yes	
$l_P := l_P + l$	$l \in \mathbb{N}$	yes	
$l := l + l_P$	$l \in \mathbb{N}$	no	$l = l' = 1$
$l_P := c$	$l, c \in \mathbb{N}$	yes	

Table 3: Each row shows a single-instruction program, whether the program gives rise to a monotone system and, if not, an assignment that violates Eq. (17). (Some of these programs are not finite-state.)

More interesting for us is the fact that asynchronous programs (= our input language) are monotone, while their parametric predicate abstractions may not be; this demonstrates that the monotonicity is in fact *lost in the abstraction*. Consider again the decrement instruction $l := l - 1$, but this time abstracted against the inter-thread predicate $Q :: l = l_P$. Parametric abstraction results in the two-thread and three-thread template instantiations

$$\begin{aligned} \tilde{\mathcal{R}}^2 &= (\neg b_1 \vee \neg b'_1) \wedge b_1 = b_2 \wedge b'_1 = b'_2 \\ \tilde{\mathcal{R}}^3 &= (\neg b_1 \vee \neg b'_1) \wedge b_1 = b_2 = b_3 \wedge b'_1 = b'_2 = b'_3. \end{aligned}$$

Consider the transition $(0, 0) \rightarrow (1, 1) \in \tilde{\mathcal{R}}^2$ and the three-thread state $w = (0, 0, 1) \succ (0, 0) : w$ clearly has no successor in $\tilde{\mathcal{R}}^3$ (it is in fact inconsistent), violating monotonicity. We discuss in Sect. 4.2 what happens to the decrement instruction with respect to predicate $l < l_P$.

4.2 Restoring Monotonicity in the Abstraction

Our goal is now to restore the monotonicity that was lost in the parametric abstraction. The standard covering relation \preceq defined over local state counter tuples turns **monotone** and **Boolean** DR programs into instances of well quasi-ordered transition systems. Program location reachability is then decidable, even for unbounded threads.

In order to do so, we first derive a sufficient condition for monotonicity that can be checked **locally** over $\tilde{\mathcal{R}}$, as follows.

Theorem 7 *Let $\tilde{\mathcal{R}}$ be the transition relation of a DR program. Then the infinite-state transition system $\cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$ is monotone if the following formula over $\tilde{L} \times \tilde{L}'$ is valid:*

$$\exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}} \Rightarrow \forall \tilde{L}_P \exists \tilde{L}'_P : \tilde{\mathcal{R}}. \quad (17)$$

Unlike the monotonicity characterization given in Lemma 6, Eq. (17) is formulated only about the template program $\tilde{\mathcal{R}}$. It suggests that, if $\tilde{\mathcal{R}}$ holds for some valuation of its passive-thread variables, then no matter how we replace the current-state passive-thread variables \tilde{L}_P , we can find next-state passive-thread variables \tilde{L}'_P such that $\tilde{\mathcal{R}}$ still holds. This is true for asynchronous programs, since here $\tilde{L}_P = \emptyset$. It fails for the swap instruction in the first row of Table 3: the instruction gives rise to the DR program $\tilde{\mathcal{R}} :: 1' = 1_P \wedge 1'_P = 1$. The assignment on the right in the table satisfies $\tilde{\mathcal{R}}$, but if 1_P is changed to 0, $\tilde{\mathcal{R}}$ is violated no matter what value is assigned to $1'_P$.

We are now ready to modify the possibly non-monotone abstract DR program $\tilde{\mathcal{P}}$ into a new, monotone abstraction $\tilde{\mathcal{P}}_m$. Our solution is similar in spirit to, but different in effect from, earlier work on *monotonic abstractions* [3], which proposes to delete processes that violate universal guards and thus block a transition. This results in an overapproximation of the original system and thus possibly spuriously reachable error states. By contrast, exploiting the monotonicity of the *concrete* program \mathcal{P} , we can build a monotone program $\tilde{\mathcal{P}}_m$ that is safe exactly when $\tilde{\mathcal{P}}$ is, thus fully preserving soundness and precision of the abstraction $\tilde{\mathcal{P}}$.

Definition 8 *The **non-monotone fragment (NMF)** of a DR program with transition relation $\tilde{\mathcal{R}}$ is the formula over $\tilde{L} \times \tilde{L}_P \times \tilde{L}'$:*

$$\mathcal{F}(\tilde{\mathcal{R}}) :: \neg \exists \tilde{L}'_P : \tilde{\mathcal{R}} \wedge \exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}}. \quad (18)$$

The NMF encodes partial assignments $(1, 1_P, 1')$ that cannot be extended, via any $1'_P$, to a full assignment satisfying $\tilde{\mathcal{R}}$, but can be extended for some valuation of \tilde{L}_P other than 1_P . We revisit the two non-monotone instructions from Table 3. The NMF of $1, 1_P := 1_P, 1$ is $1' \neq 1_P$: this clearly cannot be extended to an assignment satisfying $\tilde{\mathcal{R}}$, but when 1_P is changed to $1'$, we can choose $1'_P = 1$ to satisfy $\tilde{\mathcal{R}}$. The non-monotone fragment of $1 := 1 + 1_P$ is $1' \geq 1 \wedge 1' \neq 1 + 1_P$.

Eq. (18) is slightly stronger than the negation of (17): the NMF binds the values of the \tilde{L}_P variables for which a violation of $\tilde{\mathcal{R}}$ is possible. It can be used to “repair” $\tilde{\mathcal{R}}$:

Lemma 9 *For a DR program with transition relation $\tilde{\mathcal{R}}$, the program with transition relation $\tilde{\mathcal{R}} \vee \mathcal{F}(\tilde{\mathcal{R}})$ is monotone.*

Lemma 9 suggests to add artificial transitions to $\tilde{\mathcal{P}}$ that allow arbitrary passive-thread changes in states of the non-monotone fragment, thus lifting the blockade previously caused by some passive threads. While this technique restores monotonicity, the problem is of course that such arbitrary changes will generally modify the program behavior; in particular, an added transition may lead a thread directly into an error state that used to be unreachable.

In order to instead obtain a *safety-equivalent* program, we prevent passive threads that block a transition in $\tilde{\mathcal{P}}^n$ from affecting the future execution. This can be realized by redirecting them to an auxiliary sink state. Let ℓ_\perp be a fresh program label.

Definition 10 *The **monotone closure** of DR program $\tilde{\mathcal{P}} = (\tilde{\mathcal{R}}, \tilde{\mathcal{I}})$ is the DR program $\tilde{\mathcal{P}}_m = (\tilde{\mathcal{R}}_m, \tilde{\mathcal{I}})$ with the transition relation $\tilde{\mathcal{R}}_m :: \tilde{\mathcal{R}} \vee (\mathcal{F}(\tilde{\mathcal{R}}) \wedge (\text{pc}'_P = \ell_\perp))$.*

This extension of the transition relation has the following effects: (i) for any program state, if any passive thread can make a move, so can all, ensuring monotonicity, (ii) the added moves do not affect the safety of the program, and (iii) transitions that were previously possible are retained, so no behavior is removed. The following theorem summarizes these claims:

Theorem 11 *Let \mathcal{P} be an asynchronous program, and $\tilde{\mathcal{P}}$ its parametric abstraction. The monotone closure $\tilde{\mathcal{P}}_m$ of $\tilde{\mathcal{P}}$ is monotone. Further, $(\tilde{\mathcal{P}}_m)^n$ is safe exactly if $\tilde{\mathcal{P}}^n$ is.*

Thm. 11 justifies our strategy for reachability analysis of an asynchronous program \mathcal{P} : form its parametric predicate abstraction $\tilde{\mathcal{P}}$ described in Sections 2 and 3, build the monotone closure $\tilde{\mathcal{P}}_m$, and analyze $(\tilde{\mathcal{P}}_m)^\infty$ using any technique for monotone systems.

Proving the parameterized ticket algorithm Applying this strategy to the ticket algorithm yields a well quasi-ordered transition system for which the backward reachability method described in [1] returns “uncoverable”, confirming that the ticket algorithm guarantees mutual exclusion, this time *for arbitrary thread counts*. We remind the reader that the ticket algorithm is challenging for existing techniques: **cream** [19], **slab** [11] and **symmpa** [10] handle only a fixed number of threads, and the resource requirements of these algorithms grow rapidly; none of them can handle even a handful of threads. The recent approach from [14] generates polynomial-size proofs, but again only for fixed thread counts.

5 Comparison with Related Work

Existing approaches for verifying asynchronous shared-memory programs typically do not exploit the monotone structure that source-level multi-threaded programs often naturally exhibit [20, 7, 9, 28, 19, 10, 12, 14]. For example, the constraint-based approach in [19], implemented in `cream`, generates Owicki-Gries and rely-guarantee type proofs. It uses predicate abstraction in a CEGAR loop to generate environment invariants for fixed thread counts, whereas our approach directly checks the interleaved state space and exploits monotonicity. Whenever possible, `cream` generates thread-modular proofs by prioritizing predicates that do not refer to the local variables of other threads.

A CEGAR approach for fixed-thread symmetric concurrent programs has been implemented in `symmpa` [10]. It uses predicate abstraction to generate a Boolean Broadcast program (a special case of DR program). Their approach cannot reason about relationships between local variables across threads, which is crucial for verifying algorithms such as the ticket lock. Nevertheless, even the restricted predicate language of [10] can give rise to non-asynchronous programs. As a result, their technique cannot be extended to unbounded thread counts with well quasi-ordered systems technology.

Recent work on data flow graph representations of fixed-thread concurrent programs has been applied to safety property verification [14]. The inductive data flow graphs can serve as succinct correctness proofs for safety properties; for the ticket example they generate correctness proofs of size quadratic in n . Similar to [14], the technique in [12] uses data flow graphs to compute invariants of concurrent programs with unbounded threads (implemented in `duet`). In contrast to our approach, which uses an expressive predicate language, `duet` constructs proofs from relationships between either solely shared or solely local variables. These are insufficient for many benchmarks such as the parameterized ticket algorithm.

Predicates that, like our inter-thread predicates, reason over all participating processes/threads have been used extensively in invariant generation methods [5, 16, 22]. As a recent example, an approach that relies on abstract interpretation instead of model checking is [25]. Starting with a set of candidate invariants (assertions), the approach builds a *reflective abstraction*, from which invariants of the concrete system are obtained in a fixed point process. These approaches and ours share the insight that complex relationships over all threads may be required to prove easy-to-state properties such as mutual exclusion. They differ fundamentally in the way these relationships are used: abstraction with respect to a given set \mathcal{Q} of quantified predicates determines the strongest invariant expressible as a Boolean formula over the set \mathcal{Q} ; the result is unlikely to be expressible in the language that defines \mathcal{Q} . Future work will investigate how invariant generation procedures can be used towards *predicate discovery* in our technique.

The idea of “making” systems monotone, in order to enable wqo-based reasoning, was pioneered in earlier work [6, 3]. Bingham and Hu deal with guards that require universal quantification over thread indices, by transforming such systems into Broadcast protocols. This is achieved by replacing conjunctively

guarded actions by transitions that, instead of checking a universal condition, execute it assuming that any thread not satisfying it “resigns”. This happens via a designated local state that isolates such threads from participation in future the computation. The same idea was further developed by Abdulla et al. in the context of *monotonic abstractions*. Our solution to the loss of monotonicity was in some way inspired by these works, but differs in two crucial aspects: first, our concrete input systems are asynchronous and thus monotone, so our incentive to *preserve* monotonicity in the abstract is strong. Second, exploiting the input monotonicity, we can achieve a monotonic abstraction that is safety-equivalent to the non-monotone abstraction and thus not merely an error-preserving approximation. This is essential, to avoid spurious counterexamples in addition to those unavoidably introduced by the predicate abstraction.

6 Concluding Remarks

We have presented in this paper a comprehensive verification method for arbitrarily-threaded asynchronous shared-variable programs. Our method is based on predicate abstraction and permits expressive universally quantified *inter-thread* predicates, which track relationships such as “my ticket number is the smallest, among all threads”. Such predicates are required to verify, via predicate abstraction, some widely used algorithms like the ticket lock. We found that the abstractions with respect to these predicates result in non-monotone finite-data replicated programs, for which reachability is in fact undecidable. To fix this problem, we strengthened the earlier method of monotonic abstractions such that it does not introduce spurious errors into the abstraction.

We view the treatment of monotonicity as the major contribution of this work. Program design often naturally gives rise to “monotone concurrency”, where adding components cannot disable existing actions, up to component symmetry. Abstractions that interfere with this feature are limited in usefulness. Our paper shows how the feature can be inexpensively restored, allowing such abstraction methods and powerful infinite-state verification methods to coexist peacefully.

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Supplemental Material

A Inter-thread Predicates are Essential for the Ticket Algorithm

Lemma 12 *Consider the parameterized ticket algorithm where threads call `spin_lock` and `spin_unlock` arbitrarily often. No Hoare/Floyd-style correctness proof over **single-thread** predicates exists.*

Proof. We write pc_i for the pc of thread i , $1 \leq i \leq n$. We first state some easy-to-prove invariants of the ticket algorithm:

$$s \leq t \leq s + n \quad (19)$$

$$\text{pc}_i = \ell_1 \Rightarrow l_i = 0 \quad (20)$$

$$\text{pc}_i = \ell_2 \Rightarrow s < l_i < t \quad (21)$$

$$\text{pc}_i = \ell_3 \Rightarrow l_i = s \quad (22)$$

$$\#(\text{pc} = \ell_2) + \#(\text{pc} = \ell_3) = t - s \quad (23)$$

We can think of ℓ_1 , ℓ_2 , and ℓ_3 as the non-critical, trying, and locked region of a standard mutex lock. The total number of threads in the trying and locked regions is $t - s$ (Eq. (23)). If all threads are “non-critical”, we have $s = t$, and the l_i are all zero.

Let now

$$E = \bigcup_{i=1}^n E^{s,t,l_i} \quad (24)$$

be the **disjoint** union of sets of predicates formulated over the shared variables s and t and any one of the l_i ; in particular, no predicate may refer to several of the l_i . Suppose I is an invariant expressible over E that is strong enough to prove mutual exclusion. Then

$$\forall i, j : i \neq j : I \wedge \text{pc}_i = \text{pc}_j = \ell_2 \Rightarrow l_i \neq l_j, \quad (25)$$

since otherwise threads i and j can, once s reaches the value $l_i (= l_j)$, escape the busy-wait loop and simultaneously proceed to the critical section.

For any c , there exists a *reachable* global state satisfying $\text{pc}_1 = \text{pc}_2 = \ell_2$ and $(s, l_1, l_2) = (c, c, c + 1)$ (that is, thread 1 proceeds to the trying region first, then thread 2), and a *reachable* global state satisfying $\text{pc}_1 = \text{pc}_2 = \ell_2$ and $(s, l_1, l_2) = (c, c + 1, c)$ (vice versa). Since c is unbounded, there thus exist infinitely many such assignments that satisfy invariant I .

Let now $\{I_1, \dots, I_w\}$ be the cubes in the DNF representation of I . Since this set is finite, there exists a single cube I_k that satisfies both $(s, l_1, l_2) = (c, c, c + 1)$ and $(s, l_1, l_2) = (c, c + 1, c)$, for some c . We split I_k into the sub-cubes that belong to E^{s,t,l_1} , and those that belong to E^{s,t,l_2} : $I_k = I_k^1 \wedge I_k^2$; note that these sub-cube sets are disjoint (sub-cubes that refer to neither l_1 nor l_2 are apportioned to either side). Then $(s, l_1, l_2) = (c, c, c + 1)$ satisfies I_k^1 , which does not contain l_2 , so in fact $(s, l_1) = (c, c)$ satisfies I_k^1 . Symmetrically, one obtains that $(s, l_2) = (c, c)$ satisfies I_k^2 . Hence $(s, l_1, l_2) = (c, c, c)$ satisfies $I_k^1 \wedge I_k^2 = I_k$ and hence satisfies I , which contradicts Eq. (25).

B Simulating Shared Via Local Variables

We have excluded shared variables from the description of dual-reference programs to simplify the notation. This is not a restriction, as such variables can be simulated via active- and passive-thread local variables, as follows. To eliminate shared variable \mathbf{s} , we instead introduce a fresh local variable $1 \in \tilde{L}$, and replace a statement like $\mathbf{s} := 5$ by the atomic statement $1 := 5, 1_P := 5$. That is, each thread keeps a local copy of what used to be the shared variable; the semantics of passive-thread variables ensures that the values are synchronized across all threads.

C Proof of Lemma 2

Lemma 2 $(\tilde{\mathcal{P}}(n))^n$ overapproximates $\hat{\mathcal{P}}^n$: For every $n \geq 2$ and every a , $(\hat{\mathcal{R}}_a)^n \Rightarrow (\tilde{\mathcal{R}}(n)_a)^n$ and $\hat{\mathcal{I}}^n \Rightarrow (\tilde{\mathcal{I}}(n)_a)^n$.

Proof (partial). For the initial states, by equations (8), (11) and (13) the implication amounts to

$$\begin{aligned} \hat{\mathcal{I}}^n &\Rightarrow \bigwedge_{p:p \neq a} \tilde{\mathcal{I}}(n) \{ \tilde{L} \triangleright \tilde{L}_a \} \{ \tilde{L}_P \triangleright \tilde{L}_p \} \\ &= \bigwedge_{p:p \neq a} \exists \hat{L}_3, \dots, \hat{L}_n : \hat{\mathcal{I}}^n \{ \hat{L}_1 \triangleright \tilde{L} \} \{ \hat{L}_2 \triangleright \tilde{L}_P \} \{ \tilde{L} \triangleright \tilde{L}_a \} \{ \tilde{L}_P \triangleright \tilde{L}_p \} \\ &= \bigwedge_{p:p \neq a} \exists \hat{L}_3, \dots, \hat{L}_n : \hat{\mathcal{I}}^n \{ \hat{L}_1 \triangleright \tilde{L}_a \} \{ \hat{L}_2 \triangleright \tilde{L}_p \}. \end{aligned}$$

The implication holds since the initial condition \mathcal{I} is identical for all threads (1), so replacing thread ids 1 and 2 by thread ids a and p does not falsify the formula. The case of the transition relation is similar but more involved.

D Proof of Thm. 3

Theorem 3 Let $\#_{IT}$ be the number of inter-thread predicates among the $Q[i]$. Then the sequence $(\tilde{\mathcal{P}}(n))$ stabilizes at $\mathbf{b} = 4 \times \#_{IT} + 2$, i.e. for $n \geq \mathbf{b}$, $\tilde{\mathcal{P}}(n) = \tilde{\mathcal{P}}(\mathbf{b})$.

Proof. Let $Q[1], \dots, Q[m]$ be a list of predicates $\#_{IT}$ of which are inter-thread, and let $\tilde{\mathcal{R}}_\infty$ denote the formula characterizing $\bigvee_{n=1}^\infty \tilde{\mathcal{R}}_n$ (the existence of a finite encoding is guaranteed). We show that stabilization occurs at $\mathbf{b} = 2 + 4 \times \#_{IT}$, i.e., $\tilde{\mathcal{R}}_\infty \Rightarrow \tilde{\mathcal{R}}_{\mathbf{b}}$. The proof for the stabilization of $\tilde{\mathcal{I}}$ is analogous (factor 4 then reduces to 2). We first show that stabilization occurs for the special case that all predicates are inter-thread, and then argue that this value is insensitive to the number of single-thread predicates.

The proof is by induction over m . Let first $m = \#_{IT} = 1$ and $t = (\mathbf{b}_1 \mathbf{b}_2 \mathbf{b}'_1 \mathbf{b}'_2) \in \mathbb{B}^4$ be a transition in $\tilde{\mathcal{R}}_\infty$, and $\mathcal{D}_a^n :: (b[1]_a \Leftrightarrow Q[1]_a)$. Then by definition of $\tilde{\mathcal{R}}_\infty$

and Eq. (12) there exists a thread number $n \geq 2$, and a valuation v of variables $\tilde{V}_n, \tilde{V}'_n, V_n$ and V'_n such that v satisfies $(\mathcal{R}_1)^n \wedge \bigwedge_{a=1}^2 (\mathbf{b}_a \Leftrightarrow Q[1]_a \wedge \mathbf{b}'_a \Leftrightarrow Q[1]'_a) \wedge \bigwedge_{a=3}^n \mathcal{D}_a^n \wedge \mathcal{D}_a^{n'}$. Let

$$v = (\mathbf{b}_1 \dots \mathbf{b}_n \mathbf{b}'_1 \dots \mathbf{b}'_n \mathbf{s}_1 \dots \mathbf{l}_n \mathbf{s}'_1 \dots \mathbf{l}'_n)$$

be that valuation. Then there exists a number $q \in [2, 6]$, and a map $\pi : \{1, \dots, q\} \rightarrow \{1, \dots, n\}$ such that

$$(\mathbf{b}_{\pi_1} \dots \mathbf{b}_{\pi_n} \mathbf{b}'_{\pi_1} \dots \mathbf{b}'_{\pi_n} \mathbf{s}_{\pi_1} \dots \mathbf{l}_{\pi_n} \mathbf{s}'_{\pi_1} \dots \mathbf{l}'_{\pi_n})$$

satisfies $(\mathcal{R}_1)^n \wedge \mathcal{D}^n \wedge \mathcal{D}^{n'}$, namely by defining $\pi_1 = 1$, $\pi_2 = 2$, and letting π_3, \dots, π_q identify passive threads that falsify a conjunct in each of the expanded $Q[1]_1, Q[1]_2, Q[1]'_1$, and $Q[1]'_2$ (if any)*. Then by Eq. (12) t satisfies \mathcal{R}_q (and thus \mathcal{R}_6).

For the inductive step from m to $m+1$ predicates (all inter-thread), we extend π by (at most) 4 elements. It follows that stabilization occurs at $\mathbf{b} = 2 + 4 \times \#_{IT}$ for any $m = \#_{IT} \geq 1$.

It remains to show that stabilization is not thwarted by the existence of single-thread predicates. By Eq. (3) it follows that the truth of such predicates depends only on the variables in V_n that are visible by the thread it is evaluated over, hence on variables V_2 and V'_2 for any transition in $\tilde{\mathcal{R}}_\infty$. Now observe that these values are maintained in the permutation $(\mathbf{b}_{\pi_1} \dots)$ defined above ($\pi_j = j$ for $j \leq 2$), which gives the desired result.

E Proof of Thm. 5

Theorem 5 *Program location reachability for Boolean DR programs run by an unbounded number of threads is undecidable.*

Proof (sketch). By reducing the halting problem for the Turing-powerful deterministic 2-counter Minsky machines [24] with k control states, to the reachability problem in DR programs with 3 program locations and a local variable with k values. We demonstrate the reduction using a deterministic Minsky machine that enumerates pairs in \mathbb{N}^2 (Fig. 2; the formalism is from [26]).

The machine consists of five control states $0, \dots, 4$ ($0 = \text{initial}$), two natural-number counters c_1 and c_2 (initially 0), and increment, decrement, and zero-test operations, denoted by c_i++ , c_i-- and $c_i \stackrel{?}{=} 0$, respectively. Each operation changes the control state and counter value as indicated in the figure (the decrement and zero-test operation block if c is zero and non-zero, respectively).

Control states are encoded in local variables of $\tilde{\mathcal{P}}$ ranging over $\{0, 1, 2, 3, 4\}$; as can be seen from the figure, these local variables are synchronized across the threads, so they simulate a single shared variable that tracks the control state

*Recall that according to eq. (3), $Q[1]_a$ evaluates to false whenever the following holds:

$$\exists p \neq a : \neg Q[1]_a \{L \triangleright L_a\} \{L_P \triangleright L_p\}.$$

(see App. B). Counters are encoded in program locations $\{d_0, d_1, d_2\}$ of the DR program $\tilde{\mathcal{P}}$ such that the counter value c_i equals the number of threads in location d_i , for $i \in \{1, 2\}$. Location d_0 is the single initial program location, thus with an unbounded number of threads; it merely serves as thread-pool. Control state changes turn into synchronized local variable updates, together with the following program counter modifications: for c_i++ and c_i-- a thread moves from d_0 to d_i and vice versa, respectively, and for $c_i \stackrel{?}{=} 0$ a thread in d_0 tests for the absence of a passive thread in location d_i .

Let finally ℓ_e be a special program location of $\tilde{\mathcal{P}}$ that is reached if and only if a local variable has the value that encodes the Minsky machine's halting state. The machine halts if and only if program location ℓ_e is reached in $\tilde{\mathcal{P}}$.

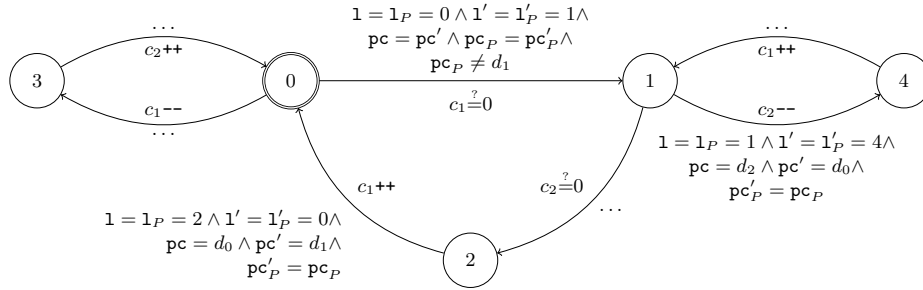


Fig. 2: Minsky machine and (part of) its DR program encoding, shown as labels of control transitions. The initial state $\tilde{\mathcal{I}}$ of the DR encoding is $1 = 1_P = 0 \wedge \text{pc} = \text{pc}_P = d_0$

Note how, in the reduction, the zero test affects the passive threads: in the transition from 0 to 1 in Fig. 2, the test on variable c_1 is simulated by asserting the absence of a passive thread in location d_1 .

F Proof of Lemma 6

Lemma 6 *Let $\tilde{\mathcal{R}}$ be the transition relation of a DR program. Then the infinite-state transition system $\cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$ is monotone (with respect to \preceq) exactly if, for all $k \geq 2$:*

$$(v, v') \in \tilde{\mathcal{R}}^k \Rightarrow \forall l_{k+1} \exists l'_{k+1}, \pi : (\langle v, l_{k+1} \rangle, \pi(\langle v', l'_{k+1} \rangle)) \in \tilde{\mathcal{R}}^{k+1} . \quad (26)$$

Proof.

“ \Rightarrow ”: suppose $\cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$ is monotone. Let $v = (l_1, \dots, l_k)$, $v' = (l'_1, \dots, l'_k)$ with $(v, v') \in \tilde{\mathcal{R}}^k$, and $w = \langle v, l_{k+1} \rangle$. We have $v \preceq w$, hence by the monotonicity of $\cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$ there exists w' such that (a) $(w, w') \in \cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$ and (b) $v' \preceq w'$. From (a) we conclude that in fact $(w, w') \in \tilde{\mathcal{R}}^{k+1}$. From (b) we conclude that

w' contains k threads in local states as in v' . Let l'_{k+1} be the local state of the additional thread (not necessarily the $k+1$ st) in w' , and σ be a permutation such that $(l'_1, \dots, l'_{k+1}) = \sigma(w')$. That is, σ reorders the local states of w' such that the k local states in v' come first, l'_{k+1} comes last. With $\pi := \sigma^{-1}$, we then have

$$\begin{aligned} (\langle v, l_{k+1} \rangle, \pi(\langle v', l'_{k+1} \rangle)) &= (\langle v, l_{k+1} \rangle, \sigma^{-1}(\langle v', l'_{k+1} \rangle)) \\ &= (w, w') \in \tilde{\mathcal{R}}^{k+1}. \end{aligned}$$

“ \Leftarrow ”: suppose $(v, v') \in \cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$, say $(v, v') \in \tilde{\mathcal{R}}^k$, so we write $v = (l_1, \dots, l_k)$ and $v' = (l'_1, \dots, l'_k)$. Let further $v \preceq w$. If w has k threads, like v , then $v \preceq w$ implies $v \succeq w$: the states are symmetry equivalent, say $w = \pi(v)$, for a permutation π on $\{1, \dots, k\}$. In this case $w' := \pi(v')$ satisfies the monotonicity conditions.

If w has $k+1$ threads, then observe that w contains k threads in local states as in v ; let l_{k+1} be the local state of the additional thread (not necessarily the $k+1$ st). Let further l'_{k+1} and π be as provided in (26). With $u = \langle v, l_{k+1} \rangle$ and $u' = \pi(\langle v', l'_{k+1} \rangle)$, we get $(u, u') \in \tilde{\mathcal{R}}^{k+1}$ by (26). Since u and w contain the same local states, let σ be a permutation such that $\sigma(u) = w$. Define $w' = \sigma(u')$. Then $w' \sim u' = \pi(\langle v', l'_{k+1} \rangle) \succeq v'$, where \sim is symmetry equivalence. Further, $(u, u') \in \tilde{\mathcal{R}}^{k+1}$ implies $(\sigma(u), \sigma(u')) \in \tilde{\mathcal{R}}^{k+1}$ by symmetry, so $(w, w') \in \tilde{\mathcal{R}}^{k+1} \subseteq \cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$, demonstrating that the monotonicity conditions are satisfied.

The case that w has more than $k+1$ threads follows by induction.

G Proof of Thm. 7

Theorem 7 *Let $\tilde{\mathcal{R}}$ be the transition relation of a DR program. Then the infinite-state transition system $\cup_{n=1}^{\infty} \tilde{\mathcal{R}}^n$ is monotone if the following formula over $\tilde{L} \times \tilde{L}'$ is valid:*

$$\exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}} \Rightarrow \forall \tilde{L}_P \exists \tilde{L}'_P : \tilde{\mathcal{R}}. \quad (27)$$

Proof. We show monotonicity using Lemma 6. Suppose $(v, v') \in \tilde{\mathcal{R}}^k$, and let l_{k+1} be given. By (9), there exists $a \in \{1, \dots, k\}$ such that $(v, v') \in (\tilde{\mathcal{R}}_a)^k$. By (10), we have

$$\forall p \in \{1, \dots, k\} \setminus \{a\} \quad \tilde{\mathcal{R}}\{\tilde{L} \bowtie \tilde{L}_a\}\{\tilde{L}_P \bowtie \tilde{L}_p\}. \quad (28)$$

Since $k \geq 2$, the quantification in (28) is not empty and hence satisfies the left-hand side of (27). By the right-hand side, there exists a valuation l'_{k+1} of all \tilde{L}'_P variables such that, replacing the \tilde{L}_P variables by the valuation l_{k+1} , $\tilde{\mathcal{R}}$ still holds, i.e. $\tilde{\mathcal{R}}\{\tilde{L} \bowtie \tilde{L}_a\}\{\tilde{L}_P \bowtie \tilde{L}_{k+1}\}$. Merging this with (28), we obtain

$$\forall p \in \{1, \dots, k+1\} \setminus \{a\} \quad \tilde{\mathcal{R}}\{\tilde{L} \bowtie \tilde{L}_a\}\{\tilde{L}_P \bowtie \tilde{L}_p\},$$

and thus $(\langle v, l_{k+1} \rangle, \langle v', l'_{k+1} \rangle) \in (\tilde{\mathcal{R}}_a)^{k+1} \subset \tilde{\mathcal{R}}^{k+1}$, establishing the right-hand side of (26) with the identity permutation π .

H Proof of Lemma 9

Lemma 9 *For a DR program with transition relation $\tilde{\mathcal{R}}$, the program with transition relation $\tilde{\mathcal{R}} \vee \mathcal{F}(\tilde{\mathcal{R}})$ is monotone.*

Proof. We show that $\tilde{\mathcal{R}} \vee \mathcal{F}(\tilde{\mathcal{R}})$ satisfies (27), i.e.

$$\exists \tilde{L}_P \tilde{L}'_P : (\tilde{\mathcal{R}} \vee \mathcal{F}(\tilde{\mathcal{R}})) \Rightarrow \forall \tilde{L}_P \exists \tilde{L}'_P : (\tilde{\mathcal{R}} \vee \mathcal{F}(\tilde{\mathcal{R}})) \quad (29)$$

Monotonicity then follows using Thm. 7.

We first simplify the right-hand side of (29):

$$\begin{aligned} & \forall \tilde{L}_P \exists \tilde{L}'_P : (\tilde{\mathcal{R}} \vee (\neg \exists \tilde{L}'_P : \tilde{\mathcal{R}} \wedge \exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}})) \\ &= \forall \tilde{L}_P : (\exists \tilde{L}'_P : \tilde{\mathcal{R}} \vee (\neg \exists \tilde{L}'_P : \tilde{\mathcal{R}} \wedge \exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}})) \\ &= \forall \tilde{L}_P : (\exists \tilde{L}'_P : \tilde{\mathcal{R}} \vee \exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}}) \\ &= \forall \tilde{L}_P : (\exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}}) \\ &= \exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}}. \end{aligned}$$

Eq. (29) now becomes

$$\exists \tilde{L}_P \tilde{L}'_P : (\tilde{\mathcal{R}} \vee (\neg \exists \tilde{L}'_P : \tilde{\mathcal{R}} \wedge \exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}})) \Rightarrow \exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}}$$

which trivially reduces to *true*, in both cases of the disjunction.

I Proof of Thm. 11

Theorem 11 *Let \mathcal{P} be an asynchronous program, and $\tilde{\mathcal{P}}$ its parametric abstraction. The monotone closure $\tilde{\mathcal{P}}_m$ of $\tilde{\mathcal{P}}$ is monotone. Further, $(\tilde{\mathcal{P}}_m)^n$ is safe exactly if $\tilde{\mathcal{P}}^n$ is.*

Proof.

- (i) Monotonicity of $\tilde{\mathcal{P}}_m$: employing Thm. 7, we prove that the following formula is valid:

$$\exists \tilde{L}_P \tilde{L}'_P : \tilde{\mathcal{R}}_m \Rightarrow \forall \tilde{L}_P \exists \tilde{L}'_P : \tilde{\mathcal{R}}_m.$$

Let $(1, 1') \in \tilde{L} \times \tilde{L}'$ be arbitrary, and suppose there exist $(1_P, 1'_P) \in \tilde{L}_P \times \tilde{L}'_P$ such that $(1, 1_P, 1', 1'_P) \in \tilde{\mathcal{R}}_m$. Let further $\mathfrak{m}_P \in \tilde{L}_P$. We construct $\mathfrak{m}'_P \in \tilde{L}'_P$ such that $(1, \mathfrak{m}_P, 1', \mathfrak{m}'_P) \in \tilde{\mathcal{R}}_m$.

Since $\tilde{\mathcal{R}}_m = \tilde{\mathcal{R}} \vee (\mathcal{F}(\tilde{\mathcal{R}}) \wedge (\mathbf{pc}'_P = \ell_\perp))$, we have either $(1, 1_P, 1', 1'_P) \in \tilde{\mathcal{R}}$ or $(1, 1_P, 1', 1'_P) \in \mathcal{F}(\tilde{\mathcal{R}}) \wedge (\mathbf{pc}'_P = \ell_\perp)$. In both cases, $(1, 1_P, 1', 1'_P) \in \tilde{\mathcal{R}} \vee \mathcal{F}(\tilde{\mathcal{R}})$. The latter relation is monotone by Lemma 9. Hence there exists some \mathbf{k}'_P such that $(1, \mathfrak{m}_P, 1', \mathbf{k}'_P) \in \tilde{\mathcal{R}} \vee \mathcal{F}(\tilde{\mathcal{R}})$.

This element \mathbf{k}'_P is almost the element $\mathfrak{m}'_P \in \tilde{L}'_P$ we are looking for: if we have $(1, \mathfrak{m}_P, 1', \mathbf{k}'_P) \in \tilde{\mathcal{R}} \subset \tilde{\mathcal{R}}_m$, then the choice $\mathfrak{m}'_P = \mathbf{k}'_P$ ensures $(1, \mathfrak{m}_P, 1', \mathfrak{m}'_P) \in \tilde{\mathcal{R}}_m$. Otherwise $(1, \mathfrak{m}_P, 1', \mathbf{k}'_P) \in \mathcal{F}(\tilde{\mathcal{R}})$. Formula $\mathcal{F}(\tilde{\mathcal{R}})$ does not contain \tilde{L}'_P variables, however; the latter can thus be replaced freely without affecting membership in $\mathcal{F}(\tilde{\mathcal{R}})$. Let therefore $\mathfrak{m}'_P :: (\exists \mathbf{pc}'_P \mathbf{k}'_P) \wedge (\mathbf{pc}'_P = \ell_\perp)$. The latter expression denotes the replacement of the value of \mathbf{pc}'_P in \mathbf{k}'_P by ℓ_\perp . Now we have $(1, \mathfrak{m}_P, 1', \mathfrak{m}'_P) \in \mathcal{F}(\tilde{\mathcal{R}})$ and in fact $(1, \mathfrak{m}_P, 1', \mathfrak{m}'_P) \in \mathcal{F}(\tilde{\mathcal{R}}) \wedge (\mathbf{pc}'_P = \ell_\perp) \subset \tilde{\mathcal{R}}_m$.

- (ii) Safety equivalence: from Def. 10 (applied to $\tilde{\mathcal{P}}$) we conclude $\tilde{\mathcal{R}} \Rightarrow \tilde{\mathcal{R}}_m$ is valid, and thus every execution of $\tilde{\mathcal{P}}$ is an execution of $\tilde{\mathcal{P}}_m$. Thus if $\tilde{\mathcal{P}}_m$ is safe, so is $\tilde{\mathcal{P}}$.

For the converse argument observe that every infinite trace π of $\tilde{\mathcal{P}}_m$ gives rise to a sequence of j traces of $\tilde{\mathcal{P}}$ as follows:

$$\pi = t_1, \dots, r_1, t_2, \dots, r_2, \dots, t_j, \dots$$

such that for all i , subtrace t_i, \dots, r_i is pairwise related by $\tilde{\mathcal{R}}$, $(r_i, t_{i+1}) \notin \tilde{\mathcal{R}}$, yet $(r_i, t_{i+1}) \in \tilde{\mathcal{R}}_m$. (If π is finite it is of the form $t_1, \dots, r_1, \dots, t_j, \dots, r_j$; the following remains valid.)

Call a state *safe* if it has no emanating execution ending in an error state. Since the asynchronous input program \mathcal{P} is monotone (“fewer threads can do less”), state safety is \prec -closed for \mathcal{P} : if a state r is safe in \mathcal{P} and $s \prec r$ then s is also safe. In order to see that the same is true for the (possibly non-monotone) abstract DR program $\tilde{\mathcal{P}}$, let R be the concretization of a state r of $\tilde{\mathcal{P}}$, i.e. a set of programs states of input program \mathcal{P} . Then $\tilde{\mathcal{P}}$ ’s conservativeness (Sect. 2.2 and Cor. 4) guarantee the safety of states in R , and \prec -closedness of state safety in \mathcal{P} implies the safety of states in the \prec -downward closure of R . From the fact that s ’s concretization is in that closure we can conclude that state safety is also \prec -downward closed for $\tilde{\mathcal{P}}$.

Using the previous result we next show that if a subtrace t_i, \dots, r_i of π contains no error state, then neither does t_{i+1}, \dots, r_{i+1} ; induction then gives us the desired result. t_1, \dots, r_1 contains no error state (otherwise $\tilde{\mathcal{P}}$ cannot be safe). The proof of the induction step is by contradiction. Assume t_i, \dots, r_i contains no error state, yet t_{i+1}, \dots, r_{i+1} does so. Let r'_i be a state such that $r'_i \prec r_i$ and $(r'_i, t_{i+1}) \in \tilde{\mathcal{R}}$. Such a state is always guaranteed to exist.* Hence r_i is safe, $r'_i \prec r_i$, yet r'_i not safe, which contradicts the property that state safety is \prec -closed for $\tilde{\mathcal{P}}$ and gives the desired result.

J Backward Reachability Tree for the Ticket Algorithm

Fig. 3 shows the backward reachability tree for the Ticket algorithm obtained using the **breach** infinite-state model checker [21].

*Such a state can always be obtained from r_i by removing the threads that were redirected to an auxiliary state in transition $(r_i, t_{i+1}) \in \tilde{\mathcal{R}}_m$.

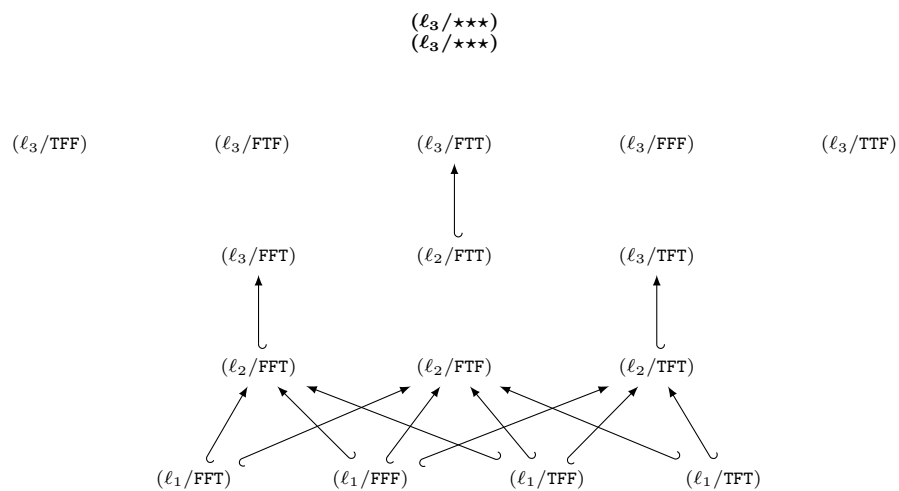


Fig. 3: The algorithm used [21] attempts to prove uncoverability of *smaller* (\prec) undecided elements first, which is why some (larger) elements are not expanded